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The Christoffel Function for Orthogonal Polynomials on a Circular Arc

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For the special type of weight functions on circular arc we study the asymptotic behavior of the Christoffel kernel off the arc and of the Christoffel function inside the arc. We prove Totik's conjecture for the Christoffel function corresponding to such weight functions.

Key Words: orthogonal polynomials on circular arc; Christoffel function; approximation.

1. BERNSTEIN–SZEGŐ METHOD FOR CIRCULAR ARC

This note sides with the paper [3], where we have developed the Bernstein–Szegő method for orthogonal polynomials with respect to the "Chebyshev-type" weight functions

$$W(e^{e\vartheta}) = \rho(e^{e\vartheta}) \frac{\sin\dfrac{\vartheta}{2}}{\sqrt{\cos^2\dfrac{\alpha}{2} - \cos^2\dfrac{\vartheta}{2}}},$$

$$\rho \in C(\Delta_\alpha), \qquad 0 < l \leqslant \rho(e^{i\vartheta}) \leqslant L < \infty \tag{1}$$

(cf. (G.4)) on the circular arc

$$\Delta_\alpha \overset{\text{def}}{=} \{e^{i\vartheta}: \alpha \leqslant \vartheta \leqslant 2\pi - \alpha\}, \qquad 0 < \alpha < \pi \tag{2}$$

(see also [1]). In what follows, formula (X) from [3] will be referred to as (G.X).

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It is well known (cf., e.g., [4, Section 3.5]) that the approximation of an arbitrary weight function W by special weight functions W_q, which makes it possible to examine the orthonormal polynomials $\varphi_n(z; W)$ by comparing them to orthonormal polynomials $\varphi_n(z; W_q)$, constitutes the key idea of the Bernstein–Szegő method. Here $q = q(n)$ depends on n and increases unboundedly as $n \to \infty$. This method has been implemented in [3, Section 4] to study the asymptotic behavior of the orthonormal polynomials φ_n with respect to the Chebyshev-type weight function W. The approximating sequence W_q takes now the form

$$W_q(e^{i\vartheta}) = \frac{\sin\dfrac{\alpha}{2}}{2\sin\dfrac{\vartheta}{2}\sqrt{\cos^2\dfrac{\alpha}{2} - \cos^2\dfrac{\vartheta}{2}}\;|\Omega_q(e^{i\omega})|^2}, \qquad e^{i\vartheta} = h(e^{i\omega}), \tag{3}$$

where h maps conformally the unit disk onto the domain $\mathbb{C}\backslash\Delta_\alpha$ (see (G.6)) and Ω_q are rational functions of special form (see [3, Section 2, Example 4]). The corresponding orthonormal polynomials $\varphi_n(z; W_q)$ named *Akhiezer's polynomials* are known explicitly for $n \geqslant q+1$:

$$\varphi_n(z; W_q) = C_n \left\{ \frac{\Omega_q\left(\dfrac{1}{v}\right)}{1-\beta v}\, w^n(v) + \frac{v\Omega_q(v)}{v-\beta}\, w^n\left(\frac{1}{v}\right) \right\},$$

$$|C_n|^2 = \frac{2\sin\dfrac{\alpha}{2}}{1+\sin\dfrac{\alpha}{2}}. \tag{4}$$

Here $z = h(v)$ and

$$w(v) \stackrel{\text{def}}{=} i\,\frac{1-\beta v}{v+\beta}, \qquad \beta = i\tan\frac{\pi-\alpha}{4}.$$

It has been proved in [3, Section 4] that the sequence Ω_q is uniformly bounded on Δ_α (cf. (G.63)) and

$$\rho_q(e^{i\vartheta}) \stackrel{\text{def}}{=} \frac{\sin\dfrac{\alpha}{2}}{2\sin^2\dfrac{\vartheta}{2}\,|\Omega_q(e^{i\omega})|^2} = W_q(e^{i\vartheta})\,\frac{\sqrt{\cos^2\dfrac{\alpha}{2} - \cos^2\dfrac{\vartheta}{2}}}{\sin\dfrac{\vartheta}{2}}$$

(resp. $\rho_q^{-1}(e^{i\vartheta})$) converges to

$$\rho(e^{i\vartheta}) \overset{\text{def}}{=} W(e^{i\vartheta}) \frac{\sqrt{\cos^2 \frac{\alpha}{2} - \cos^2 \frac{\vartheta}{2}}}{\sin \frac{\vartheta}{2}}$$

(resp. $\rho^{-1}(e^{i\vartheta})$) uniformly on Δ_α,

$$\lim_{n\to\infty} \|\rho_q^{\pm 1}(e^{i\vartheta}) - \rho^{\pm 1}(e^{i\vartheta})\|_\infty = 0, \tag{5}$$

as long as $\lim_{n\to\infty} q(n) = \infty$ (cf. [3, Section 4, Remark 5]). Moreover

$$\lim_{n\to\infty} \left\| 1 - \frac{W_q(e^{i\vartheta})}{W(e^{i\vartheta})} \right\|_\infty = \lim_{n\to\infty} \left\| 1 - \frac{W(e^{i\vartheta})}{W_q(e^{i\vartheta})} \right\|_\infty = 0 \tag{6}$$

(cf. (G.67)–(G.68)). Here and in what follows $\|\cdot\|_\infty$ stands for the uniform norm on Δ_α.

2. ASYMPTOTICS FOR CHRISTOFFEL FUNCTION

Asymptotics off the Arc Δ_α

We begin with the asymptotic behavior of the so-called *Christoffel kernel*

$$K_n(z, u; W) \overset{\text{def}}{=} \sum_{k=0}^{n-1} \varphi_k(z)\, \overline{\varphi_k(u)}$$

uniformly inside $\mathbb{C}\backslash\Delta_\alpha$.

According to [3, Theorem 6] for the orthonormal polynomials φ_n the limit relation

$$\varphi_n(z) = F(z)\, g(z)\, w^n(z)(1 + o(1)), \qquad n \to \infty,$$

holds uniformly inside $\mathbb{C}\backslash\Delta_\alpha$ with

$$F(z) = \frac{z - 1 - 2\sin\frac{\alpha}{2} + \sqrt{(z+1)^2 - 4\gamma^2 z}}{2\sqrt{1 + \sin\frac{\alpha}{2}\,(z-1)}},$$

$$w(z) = \frac{z + 1 + \sqrt{(z+1)^2 - 4\gamma^2 z}}{2\gamma} \tag{7}$$

(see (G.13), (G.15)). Here $g(z)$ is the outer function for Δ_α with the boundary values $\rho(e^{i\vartheta})$ (cf. (G.34)). For the reversed $*$-polynomials φ_n^* we have then

$$\varphi_n^*(z)=z^n\overline{\varphi_n(1/\bar{z})}=z^n\overline{F(1/\bar{z})\;g(1/\bar{z})\;w^n(1/\bar{z})}(1+o(1)).$$

We can now find the limit value for the Christoffel kernel by using the Christoffel–Darboux formula (cf. [4, p. 41, formula (1)])

$$K_n(z,u;\,W)=\frac{\varphi_n^*(z)\,\overline{\varphi_n^*(u)}-\varphi_n(z)\,\overline{\varphi_n(u)}}{1-z\bar{u}}.$$

The symmetry property (G.7) leads to (see (G.34))

$$\overline{g(1/\bar{z};\,\rho)}=\overline{\tilde{g}(1/\bar{v};\,\tilde{\rho})}=\frac{1}{\tilde{g}(v;\,\tilde{\rho})}$$

$$=\frac{1}{g(z;\,\rho)},\qquad \tilde{\rho}(e^{i\omega})=\rho(h(e^{i\omega})),\quad \tilde{g}(v,\tilde{\rho})=g(z,\rho).$$

Next, it follows from (G.15) that $\overline{w(1/\bar{z})}=z^{-1}w(z)$. Hence

$$\varphi_n^*(z)\,\overline{\varphi_n^*(u)}=\frac{\overline{F(1/\bar{z})}\,F(1/\bar{u})}{g(z;\,\rho)\,\overline{g(u;\,\rho)}}\,w^n(z)\,\overline{w^n(u)}(1+o(1)),\qquad n\to\infty$$

uniformly inside $\mathbb{C}\backslash\Delta_\alpha$. Thereby the following result has been verified.

THEOREM 1. *For the Christoffel kernel, corresponding to the Chebyshev-type weight function W the limit relation*

$$\lim_{n\to\infty}\frac{K_n(z,u;\,W)}{w^n(z)\,\overline{w^n(u)}}=\frac{1}{1-z\bar{u}}\left(\frac{\overline{F(1/\bar{z})}\,F(1/\bar{u})}{g(z;\,\rho)\,\overline{g(u;\,\rho)}}-F(z)\,\overline{F(u)}\,g(z;\,\rho)\,\overline{g(u;\,\rho)}\right)$$

holds uniformly inside $(\mathbb{C}\backslash\Delta_\alpha)\times(\mathbb{C}\backslash\Delta_\alpha)$. *Here F, w are defined in* (7) *and g is the outer function for* Δ_α *with the boundary values* $\rho(e^{i\vartheta})$.

Asymptotics on the Arc Δ_α

The main result on the present paper concerns the asymptotic behavior of the *Christoffel function*

$$\omega_n(e^{i\vartheta};\,W)\overset{\text{def}}{=}K_n^{-1}(e^{i\vartheta};\,W),$$

$$K_n(e^{i\vartheta};\,W)\overset{\text{def}}{=}K_n(e^{i\vartheta},e^{i\vartheta};\,W)=\sum_{k=0}^{n-1}|\varphi_k(e^{i\vartheta})|^2 \tag{8}$$

inside the arc Δ_α.

It is well known that the Christoffel kernel

$$K_n(z, e^{i\vartheta}; W) = \sum_{k=0}^{n-1} \varphi_k(z)\, \overline{\varphi_k(e^{i\vartheta})}$$

provides a unique solution of the extremal problem

$$\frac{1}{K_n(e^{i\vartheta}; W)} = \min \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \left| \frac{P(e^{it})}{P(e^{i\vartheta})} \right|^2 W(e^{it})\, dt$$
$$= \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \left| \frac{K_n(e^{it}, e^{i\vartheta}; W)}{K_n(e^{i\vartheta}; W)} \right|^2 W(e^{it})\, dt,$$

where minimum is taken over all polynomials P of degree less than n, which satisfy $P(e^{i\vartheta}) \neq 0$. Therefore

$$K_n^{-1}(e^{i\vartheta}; W) \leqslant \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \left| \frac{K_n(e^{it}, e^{i\vartheta}; W_q)}{K_n(e^{i\vartheta}; W_q)} \right|^2 W(e^{it})\, dt$$
$$= K_n^{-1}(e^{i\vartheta}; W_q) + \frac{1}{2\pi} \int_\alpha^{2\pi-\alpha} \left| \frac{K_n(e^{it}, e^{i\vartheta}; W_q)}{K_n(e^{i\vartheta}; W_q)} \right|^2$$
$$\times W_q(e^{it}) \left(\frac{W}{W_q} - 1 \right) dt$$
$$\leqslant K_n^{-1}(e^{i\vartheta}; W_q) \left\{ 1 + \left\| \frac{W}{W_q} - 1 \right\|_\infty \right\}.$$

In exactly the same way we obtain

$$K_n^{-1}(e^{i\vartheta}; W_q) \leqslant K_n^{-1}(e^{i\vartheta}; W) \left\{ 1 + \left\| \frac{W_q}{W} - 1 \right\|_\infty \right\}.$$

Thus the inequalities

$$\frac{\left\| \frac{W}{W_q} - 1 \right\|_\infty}{1 + \left\| \frac{W}{W_q} - 1 \right\|_\infty} \leqslant \frac{n^{-1} K_n(e^{i\vartheta}; W)}{n^{-1} K_n(e^{i\vartheta}; W_q)} - 1 \leqslant \left\| \frac{W_q}{W} - 1 \right\|_\infty \tag{9}$$

hold and due to the approximation properties (6) we see that

$$\lim_{n \to \infty} \left\| \frac{n^{-1} K_n(e^{i\vartheta}; W)}{n^{-1} K_n(e^{i\vartheta}; W_q)} - 1 \right\|_\infty = 0.$$

Our immediate objective is to compute the limit

$$\lim_{n\to\infty}\frac{1}{n}K_n(e^{i\vartheta};W_q)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\varphi_k(e^{i\vartheta};W_q)|^2,$$

which holds uniformly inside Δ_α, where $q=q(n)$ increases unboundedly in a special way as $n\to\infty$. By (4) we have for $p\geqslant q+1$,

$$|C_p|^{-2}|\varphi_p(e^{i\vartheta};W_q)|^2=|\Omega_q(e^{i\omega})|^2\{|e^{i\omega}-\beta|^{-2}+|1-\beta e^{i\omega}|^{-2}\}$$
$$+2\Re\left\{\frac{e^{i\omega}\Omega_q(e^{i\omega})\,\overline{\Omega_q(e^{-i\omega})}}{(e^{i\omega}-\beta)(1+\beta e^{-i\omega})}w^p(e^{-i\omega})\,\overline{w^p(e^{i\omega})}\right\}.$$

Put

$$\frac{e^{i\omega}\Omega_q(e^{i\omega})\,\overline{\Omega_q(e^{-i\omega})}}{(e^{i\omega}-\beta)(1+\beta e^{-i\omega})}=S_q(e^{i\omega})+iT_q(e^{i\omega})$$

and (see (G.12))

$$w^p(e^{-i\omega})\,\overline{w^p(e^{i\omega})}=\frac{e^{i\omega}-\beta}{1+\beta e^{i\omega}}\,\frac{e^{i\omega}+\beta}{1-\beta e^{i\omega}}=e^{i\tau_-}e^{i\tau_+}=e^{i\tau},$$

so that

$$|\varphi_p(e^{i\vartheta};W_q)|^2=|C_p\Omega_q(e^{i\omega})|^2\{|e^{i\omega}-\beta|^{-2}+|1-\beta e^{i\omega}|^{-2}\}$$
$$+2|C_p|^2(S_q(e^{i\omega})\cos p\tau-T_q(e^{i\omega})\sin p\tau).$$

If we sum up these equalities from $q+1$ to $n-1$, we come to the relation

$$\sum_{p=q+1}^{n-1}|\varphi_p(e^{i\vartheta};W_q)|^2$$
$$=\frac{2\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}(n-q-1)\,|\Omega_q(e^{i\omega})|^2\{|e^{i\omega}-\beta|^{-2}+|1-\beta e^{i\omega}|^{-2}\}$$
$$+\frac{4\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}\sum_{p=q+1}^{n-1}(S_q(e^{i\omega})\cos p\tau-T_q(e^{i\omega})\sin p\tau).\tag{10}$$

It can be easily seen that $\tau=0$ if and only if $\theta=\alpha$ or $\theta=2\pi-\alpha$, which corresponds to the endpoints of the arc Δ_α. Thus for a compact set $\Delta\subset\Delta_\alpha^0$, the relations

$$\sum_{p=q+1}^{n-1}\cos p\tau=O(1),\qquad \sum_{p=q+1}^{n-1}\sin p\tau=O(1),\qquad n\to\infty$$

are true uniformly on Δ.

Next, we can compute directly

$$|e^{i\omega}-\beta|^2=1+|\beta|^2-2\Re(e^{i\omega}\bar{\beta})$$

$$=1+\tan^2\eta-2\sin\omega\tan\eta=\frac{1-\sin\omega\cos\dfrac{\alpha}{2}}{\cos^2\eta},$$

$$|1-\beta e^{i\omega}|^2=1+|\beta|^2-2\Re(e^{i\omega}\beta)$$

$$=1+\tan^2\eta+2\sin\omega\tan\eta=\frac{1+\sin\omega\cos\dfrac{\alpha}{2}}{\cos^2\eta}$$

and (cf. (G.9))

$$|e^{i\omega}-\beta|^{-2}+|1-\beta e^{i\omega}|^{-2}=\frac{2\cos^2\eta}{1-\sin^2\omega\cos^2\dfrac{\alpha}{2}}=\frac{1+\sin\dfrac{\alpha}{2}}{\sin^2\dfrac{\alpha}{2}}\sin^2\frac{\vartheta}{2}.$$

It follows now from (10) that

$$\sum_{p=q+1}^{n-1}|\varphi_p(e^{i\vartheta};W_q)|^2=(n-q-1)\frac{2\sin^2\dfrac{\vartheta}{2}}{\sin\dfrac{\alpha}{2}}|\Omega_q(e^{i\omega})|^2+O(1)$$

$$=(n-q-1)\,\rho^{-1}(e^{i\vartheta};\Omega_q)+O(1),\qquad n\to\infty.$$

Assume that $q(n)=o(n)$, $n\to\infty$. By the limit relation (5) we have

$$\frac{1}{n}\sum_{p=q+1}^{n-1}|\varphi_p(e^{i\vartheta};W_q)|^2=\rho^{-1}(e^{i\vartheta})=\frac{\sin\theta}{\sqrt{\cos^2\dfrac{\alpha}{2}-\cos^2\dfrac{\vartheta}{2}}}\frac{1}{W(e^{i\vartheta})}\qquad(11)$$

uniformly inside Δ_α.

Our next goal is to show that the first q polynomials φ_n exert no influence on the asymptotic behavior of the Christoffel function as long as $q=q(n)$ goes to infinity slow enough.

The orthonormal polynomials on the unit circle satisfy the Szegő recurrences

$$\kappa_{m-1}\varphi_m(z)=z\kappa_m\varphi_{m-1}(z)+\varphi_m(0)\,\varphi^*_{m-1}(z),$$
$$\varphi_m(z)=\kappa_m z^m+\cdots,\qquad \kappa_m>0$$

(cf. [5, formula (11.4.7), p. 293]). From the relations

$$\kappa_m^{-2}=\prod_{k=1}^{m}(1-|\Phi_k(0)|^2),\qquad m=1,2,...$$

(cf. [2, p. 7]), where we write $\Phi_m(z)=\kappa_m^{-1}\varphi_m(z)$ for the corresponding monic orthogonal polynomials it follows that

$$|\kappa_{m-1}\varphi_m(e^{i\vartheta})|\geqslant\kappa_m(1-|\Phi_m(0)|)\,|\varphi_{m-1}(e^{i\vartheta})|,$$
$$|\varphi_m(e^{i\vartheta})|\geqslant\frac{\kappa_m}{\kappa_{m-1}}(1-|\Phi_m(0)|)\,|\varphi_{m-1}(e^{i\vartheta})|$$
$$\geqslant\frac{\kappa_m}{\kappa_{m-j}}\prod_{k=m-j+1}^{m}(1-|\Phi_k(0)|)\,|\varphi_{m-j}(e^{i\vartheta})|,$$

and

$$|\varphi_{m-j}(e^{i\vartheta})|\leqslant|\varphi_m(e^{i\vartheta})|\prod_{k=m-j+1}^{m}\left(\frac{1+|\Phi_k(0)|}{1-|\Phi_k(0)|}\right)^{1/2}$$
$$\leqslant 2^j\,|\varphi_m(e^{i\vartheta})|\prod_{k=m-j+1}^{m}(1-|\Phi_k(0)|^2)^{-1/2}$$
$$=2^j\frac{\kappa_m}{\kappa_{m-j}}\,|\varphi_m(e^{i\vartheta})|.$$

Finally,

$$|\varphi_p(e^{i\vartheta})|\leqslant 2^m\frac{\kappa_m}{\kappa_0}\,|\varphi_m(e^{i\vartheta})|,\qquad p=0,1,...,m. \tag{12}$$

We will apply (12) for the Akhiezer orthonormal polynomials (4) and $m=q(n)+1$. First, due to the boundedness of Ω_q it is clear that

$$\|\varphi_m(e^{i\vartheta},W_q)\|_\infty=O(1),\qquad n\to\infty. \tag{13}$$

Next, the relations (G.41) and (5) imply

$$\kappa_m(W_q)\,\gamma^m = O(1), \qquad n \to \infty, \quad \gamma = \cos\frac{\alpha}{2}.$$

Then

$$\kappa_0^{-2}(W_q) = \int_\alpha^{2\pi-\alpha} W_q(e^{i\vartheta})\,d\vartheta$$

$$= \int_\alpha^{2\pi-\alpha} \rho_q(e^{i\vartheta}) \frac{\sin\frac{\vartheta}{2}}{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\vartheta}{2}}}\,d\vartheta = O(1), \qquad n \to \infty.$$

Finally, we conclude that

$$\|\varphi_p(e^{i\vartheta}; W_q)\|_\infty \leqslant C\left(\frac{2}{\gamma}\right)^q, \qquad p = 0, 1, ..., q. \tag{14}$$

Going back to the Christoffel function, we have by (14)

$$\frac{1}{n}\sum_{p=0}^{q} |\varphi_p(e^{i\vartheta}; W_q)|^2 \leqslant C\,\frac{q+1}{n}\left(\frac{2}{\gamma}\right)^{2q}.$$

Assume that

$$q\exp\left(2q\log\frac{2}{\gamma}\right) = o(n), \qquad n \to \infty$$

(we can put, e.g., $q(n) = (\log n)^{1/2}$). In this case

$$\lim_{n\to\infty} \frac{1}{n}\sum_{p=0}^{q} |\varphi_p(e^{i\vartheta}; W_q)|^2 = 0$$

uniformly inside Δ_α. We can now combine the latter relation with (11) to establish our main result.

THEOREM 2. *For the Christoffel function ω_n (8) corresponding to Chebyshev type weight function W the limit relation*

$$\lim_{n\to\infty} n\omega_n(e^{i\vartheta}; W) = W(e^{i\vartheta}) \frac{\sqrt{\cos^2\frac{\alpha}{2} - \cos^2\frac{\vartheta}{2}}}{\sin\frac{\vartheta}{2}} \tag{15}$$

holds uniformly inside Δ_α.

Remark 3. The right-hand side in (15) can be easily recognized as the ratio

$$W(e^{i\vartheta})\,\frac{\sqrt{\cos^2\dfrac{\alpha}{2}-\cos^2\dfrac{\vartheta}{2}}}{\sin\dfrac{\vartheta}{2}}=\frac{W(e^{i\vartheta})}{p(e^{i\vartheta})},$$

where $p(e^{i\vartheta})$ is the density of the equilibrium measure for Δ_α. Thus Totik's conjecture (cf. [6, p. 217]) concerning the asymptotic behavior of the Christoffel function has been confirmed for orthogonal polynomials on the circular arc with respect to Chebyshev-type weight functions.

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